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SMALL PERTURBATIONS AND STOCHASTIC GAMES

BY

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ABSTRACT

The purpose of this note is to apply results from the theory of Markov chains with rare transitions to stochastic games. The results obtained here are used in the proof of existence of equilibrium payoffs in two-player stochastic games.

Introduction

The main purpose of this note is to apply results from the theory of Markov chains with rare transitions to stochastic games. These results are briefly introduced in the beginning of Section 1. Section 1 also contains additional results. Applications to stochastic games are given in Section 2. These results are extensively used in [10].

Throughout the paper, we use $\subseteq$ and $\subset$ for weak and strict inclusion respectively.

1. Markov chains

Most results in Section 1.1 are fairly standard. We refer to [1] for a detailed survey. Our presentation differs in two respects. First, our setup differs from the usual Markov chain setup in which the transition is assumed irreducible. Second,

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the literature on Markov chains with rare transitions assumes that the transitions satisfy a large deviation principle. We make no such assumption. Even though our treatment eventually turns out not to introduce additional generality (see Section 1.3), it allows one to define a convenient compactification of a set of transition functions.

1.1 INTRODUCTION.

Let S be a finite set and Γ be a directed graph on S. We let S^* denote the set of sinks for Γ, i.e., the set of vertices $s \in S$, such that no edge is incident out of s. We assume that $S^* \neq \emptyset$, and that for each $s \in S$, there exists a path from s to S^*.

Let $\mathcal{O}$ be the set of all transition probabilities p on S, such that

$$p(s'|s) > 0 \Leftrightarrow (s,s') \text{ is an edge of } \Gamma.$$

Let $(s_k)_{k\geq 1}$ be a Markov chain on S, with transition $p \in \mathcal{O}$, and initial state $s \in S$. We denote by $\mathbf{P}_{s,p}$ the law of (s_k). The assumption on Γ is equivalent to saying that the elements of S^* are absorbing states, and that $(s_k)_k$ reaches S^* in finite time, whatever s.

For $C \subseteq S\backslash S^*$, we let $e_C = \inf\{k \geq 1, s_k \notin C\}$ be the stage of first exit from C, and we let $\mathbf{Q}_{s,p}(\cdot|C)$ denote the law of s_{e_C}. It is well-defined since $e_C < +\infty$, $\mathbf{P}_{s,p}$-a.s. For $s \in C$, we denote by $r_s = \inf\{k > 1, s_k = s\}$ the first return to s.

Freidlin and Wentzell expressed $\mathbf{Q}_{s,p}(\cdot|C)$ in terms of graphs. For $C \subseteq S\backslash S^*$, define a C-graph to be a subgraph g of Γ such that

- for each $s \in C$, there is exactly one edge incident out of s; we denote by $g(s)$ its endpoint;
- for each $s \notin C$, there is no edge incident out of s;
- g has no loop.

It is clear that for every $s \in C$, there is a unique state $s' \notin C$, such that g contains a path from s to s'; moreover, this path is unique. In such a case, we say that s leads to s' in g. We denote by G_C the set of C-graphs and by $G_C(s \to s')$, $s \in C, s' \notin C$, the set of $g \in G_C$, such that s leads to s' in g.

All graphs in the paper will have S as set of vertices. Thus, we shall identify without ambiguity each graph with its set of edges. For instance, a subgraph of Γ is simply a subset of its set of edges.

Given $p \in \mathcal{O}$, and $g \in G_C$, we define the weight of g under p by

$$w_p(g) = \prod_{(s,s') \text{ edge of } g} p(s'|s).$$

By [2] (Chapter 6, Lemma 3.3), one has

$$\mathbf{Q}_{s,p}(s'|C) = \frac{\sum_{G_C(s\to s')} w_p(g)}{\sum_{G_C} w_p(g)} \tag{1}$$

for every $s \in C, s' \notin C$,

Let $\mathcal{G}$ be the (finite) set of non-empty subgraphs of Γ. In the sequel, we let a sequence $(p_n)_n$ in $\mathcal{O}$ be given, such that

$$\theta(g_1, g_2) = \lim_{n\to+\infty} \frac{w_{p_n}(g_1)}{w_{p_n(g_2)}}$$

exists, for each $g_1, g_2 \in \mathcal{G}$. We describe the asymptotic behavior of the sequence (p_n).

We denote by

$$G_C^{\min} = \{g \in G_C \text{ such that } \theta(g', g) < +\infty, \text{ for every } g' \in G_C\}$$

the set of C-graphs g such that no graph is infinitely more probable than g. Observe that $G_C^{\min} \neq \emptyset$. Indeed, let $\bar{g} \in G_C$ be such that $w_{p_n}(\bar{g}) = \max_{g\in G_C} w_{p_n}(g)$ for infinitely many values of n. Hence

$$\theta(g, \bar{g}) = \lim_{n\to\infty} \frac{w_{p_n}(g)}{w_{p_n}(\bar{g})} \leq 1,$$

for every $g \in G_C$.

Let $\bar{g} \in G_C^{\min}$. By definition, $0 \leq \theta(g, \bar{g}) < +\infty$ for every $g \in G_C$ and $\theta(g, \bar{g}) > 0$ if and only if $g \in G_C^{\min}$.

Given $s \in C, s' \notin C$, we set $G_C^{\min}(s \to t) = G_C^{\min} \cap G_C(s \to t)$.

LEMMA 1: *The following properties hold:*

1. $p^\theta = \lim_{n\to+\infty} p_n$ *exists;*
2. *for every* $C \subseteq S\backslash S^*$, $s \in C$, $t \notin C$, $\mathbf{Q}_{s,\theta}(t|C) = \lim_{n\to\infty} \mathbf{Q}_{s,p_n}(t|C)$ *exists and*

$$\mathbf{Q}_{s,\theta}(t|C) = \lim_{n\to\infty} \frac{\sum_{g\in G_C^{\min}(s\to t)} w_{p_n}(g)}{\sum_{g\in G_C^{\min}} w_{p_n}(g)}.$$

We define the numerator on the right-hand side as zero if $G_C^{\min}(s \to t) = \emptyset$. The use of the letter θ is motivated by the remark that follows.

Proof: Let $s \in S$, and $(s, \bar{s})$ be an edge of Γ. By assumption, for every s', the sequence

$$\left(\frac{p_n(s'|s)}{p_n(\bar{s}|s)}\right)_n$$

has a limit when $n \to +\infty$. Since $\sum_{s' \in S} p_n(s'|s) = 1$ for every n, the first claim follows.

Fix $g_0 \in G_C^{\min}$. Then

$$\mathbf{Q}_{s,p_n}(t|C) = \frac{\sum_{g \in G_C(s \to t)} \frac{w_{p_n}(g)}{w_{p_n}(g_0)}}{\sum_{g \in G_C} \frac{w_{p_n}(g)}{w_{p_n}(g_0)}} \to_{n \to +\infty} \frac{\sum_{g \in G_C(s \to t)} \theta(g, g_0)}{\sum_{g \in G_C} \theta(g, g_0)}$$
$$= \frac{\sum_{g \in G_C^{\min}(s \to t)} \theta(g, g_0)}{\sum_{g \in G_C^{\min}} \theta(g, g_0)},$$

where the denominator is positive since $\theta(g_0, g_0) = 1$, and finite since $g_0 \in G_C^{\min}$. ∎

In particular, $\mathbf{Q}_{s,\theta}(t|C) > 0$ if and only if $G_C^{\min}(s \to t) \neq \emptyset$. For $t \in C$, we set $\mathbf{Q}_{s,\theta}(t|C) = 0$. Observe that $\mathbf{Q}_{s,\theta}(\cdot|C)$ is a probability distribution over S.

Remark 2: Define a map $\phi: \mathcal{O} \to]0, +\infty[^{\mathcal{G} \times G}$ by

$$\phi(p)(g_1, g_2) = \frac{w_p(g_1)}{w_p(g_2)}, \quad \text{for every } p \in \mathcal{O}, \quad g_1, g_2 \in \mathcal{G}.$$

Clearly, ϕ is an homeomorphism onto its image. Define Θ as the closure of $\phi(\mathcal{O})$ (the topology on $[0, +\infty]$ is the usual one, and $[0, +\infty]^{\mathcal{G} \times G}$ is endowed with the product topology). The topological properties of Θ have been studied, along with various variations, in different setups (see [3, 6, 8] for instance). The space Θ provides a compactification of $\mathcal{O}$, which is convenient when using, for instance, fixed-point techniques (see [7, 4] for examples in stochastic games and equilibrium refinements).

1.2 Communicating Sets. For fixed n, the Markov chain with transition function p_n reaches S^* in finite time. The law of the absorbing state $s_{e_{S \setminus S^*}}$ converges to $\mathbf{Q}_{s,\theta}(\cdot|S \setminus S^*)$ as n goes to infinity. We now describe in more detail the asymptotic behavior, as n goes to infinity. Let $C \subseteq S \setminus S^*$ be a recurrent set for p^θ. Informally, if the initial state belongs to C, the (random) number of visits to any given state of C, prior to e_C, increases to ∞ as n increases to ∞. The sets which satisfy this property are called communicating for θ. Formally, we introduce the following definition.

Definition 3: Let $C \subseteq S \setminus S^*$ be nonempty. C **communicates** for θ if

$$\lim_{n \to \infty} \mathbf{P}_{s,p_n}(e_C < r_t) = 0, \quad \text{for every } s, t \in C. \tag{2}$$

We denote by $\mathcal{C}(\theta)$ the collection of communicating sets for θ. Let $C \in \mathcal{C}(\theta)$, and $t \in C$. Observe that $\lim_{n\to+\infty} \mathbf{P}_{t,p_n}(r_t < e_C) = 1$. Denote by $N_C(t) = |\{1 < k < e_C, s_k = t\}|$ the number of passages in t before e_C. Since

$$\mathbf{P}_{s,p_n}(N_C(t) \geq 2) = \mathbf{P}_{s,p_n}(r_t < e_C) \times \mathbf{P}_{t,p_n}(r_t < e_C),$$

one has $\lim_{n\to\infty} \mathbf{P}_{s,p_n}(N_C(t) \geq 2) = 1$ and, more generally,

$$\lim_{n\to+\infty} \mathbf{P}_{s,p_n}(N_C(t) \geq q) = 1, \quad \text{for each integer } q.$$

Condition (2) is equivalent to $\mathbf{Q}_{s,\theta}(t|C\backslash\{t\}) = 1$, for every $s,t \in C$, $s \neq t$. Therefore, for any $t \in C$, and $g \in G^{\min}_{C\backslash\{t\}}$, each state in $C\backslash\{t\}$ leads to t in g. More generally, the following property holds.

LEMMA 4: *Let $\emptyset \neq D \subset C$ and $s \in D$. One has*

$$\mathbf{Q}_{s,\theta}(C|D) = 1.$$

Proof: Let $s' \notin C$, and $t \in C\backslash D$. Clearly, $\mathbf{Q}_{s,\theta}(s'|D) \leq \mathbf{Q}_{s,\theta}(s'|C\backslash\{t\}) = 0$. ∎

It is convenient to introduce the union $\overline{\mathcal{C}}(\theta)$ of $\mathcal{C}(\theta)$ and of the singleton sets $\{s\}$, $s \in S\backslash S^*$. Observe that $\overline{\mathcal{C}}(\theta)$ is the collection of sets $C \subseteq S\backslash S^*$ which satisfy

$$\lim_{n\to\infty} \mathbf{P}_{s,p_n}(e_C < r_t^*) = 0, \quad \text{for every } s,t \in C,$$

where $r_t^* = \inf\{n \geq 1, s_n = t\}$ is the first passage time in t. The elements of $\bar{\mathcal{C}}(\theta)\backslash\mathcal{C}(\theta)$ are the singleton sets which are transient for p^θ.

Observe that $C_1 \cup C_2 \in \overline{\mathcal{C}}(\theta)$ as soon as $C_1 \cap C_2 \neq \emptyset$. Therefore, $\overline{\mathcal{C}}(\theta)$, ordered by inclusion, has the structure of a forest (a collection of disjoint trees).

In this structure, the sons of $C \in \overline{\mathcal{C}}(\theta)$ are the maximal elements of $\overline{\mathcal{C}}(\theta)$ that are strict subsets of C. Since $\overline{\mathcal{C}}(\theta)$ contains the singletons, the sons of C form a partition of C.

LEMMA 5: *Let $C \in \bar{\mathcal{C}}(\theta)$, and $s, s' \in C$. One has $\mathbf{Q}_{s,\theta}(\cdot|C) = \mathbf{Q}_{s',\theta}(\cdot|C)$.*

Proof: For $n \in \mathbf{N}$, and $t \notin C$, one has

$$\mathbf{P}_{s,p_n}(s_{e_C} = t) = \mathbf{P}_{s,p_n}(s_{e_C} = t, e_C < r_{s'}^*) + \mathbf{P}_{s,p_n}(e_C > r_{s'}^*) \times \mathbf{P}_{s',p_n}(s_{e_C} = t).$$

The result follows by letting n go to infinity. ∎

We simply write $\mathbf{Q}_\theta(\cdot|C)$ instead of $\mathbf{Q}_{s,\theta}(\cdot|C)$ when $C \in \bar{\mathcal{C}}(\theta)$. We provide now a characterization of communicating sets. Let $C \subseteq S\backslash S^*$ be given. Let $\mathcal{D}_C$

denote the collection of the maximal strict subsets of C that belong to $\overline{\mathcal{C}}(\theta)$. $\mathcal{D}_C$ is simply the set of sons of C if $C \in \mathcal{C}(\theta)$. The next lemma simply says that C is a communicating set if and only if the collection of its sons is closed for the transition function defined by the exit distributions.

LEMMA 6: *$C \in \mathcal{C}(\theta)$ if and only if*

$$\mathbf{Q}_\theta(C|D) = 1 \quad \text{for each } D \in \mathcal{D}_C. \tag{3}$$

Proof: (3) is a necessary condition. Assume now that (3) holds. Let $\widetilde{p}$ be the transition function on C defined by $\widetilde{p}(\cdot|s) = \mathbf{Q}_\theta(\cdot|D)$ if $s \in D$, with $D \in \mathcal{D}_C$, and let $\bar{C} \subseteq C$ be a recurrent set for $\widetilde{p}$. We shall argue that $\bar{C} \in \overline{\mathcal{C}}(\theta)$. Since $\bar{C}$ is the union of at least two sons of C, this implies $\overline{C} = C$, by definition of $\mathcal{D}_C$.

Let $t \in \bar{C}$ be given. For $s \in \bar{C}$, define $u_n(s) = \mathbf{P}_{s,p_n}(r_t^* < e_{\bar{C}})$. We need to prove that $u(s) = \lim_{n\to\infty} u_n(s) = 1$, for each $s \in \bar{C}$. Let $s \in D$, with $D \in \mathcal{D}_C$. One has

$$\begin{aligned} u_n(s) &= \mathbf{P}_{s,p_n}(r_t^* < e_{\bar{C}}) \\ &= \sum_{s' \in \bar{C}} \mathbf{Q}_{s,p_n}(s'|D) \times \mathbf{P}_{s',p_n}(r_t^* < e_C). \end{aligned}$$

By letting $n \to +\infty$, one gets

$$u(s) = \sum_{s' \in \bar{C}} \widetilde{p}(s'|s) u(s'),$$

hence u is harmonic with respect to $\widetilde{p}$. Since $\bar{C}$ is a recurrent set for $\widetilde{p}$, u is constant on C. Since $u_n(t) = 1$ for each n, the result follows. ∎

A useful by-product of the previous proof is the following.

LEMMA 7: *Let $C \in \mathcal{C}(\theta)$, and let $\mathcal{D}_C$ be the collection of its sons. Then C is a recurrent set for the transition function $\widetilde{p}$ defined on C by $\widetilde{p}(\cdot|s) = \mathbf{Q}_\theta(\cdot|D)$ if $s \in D$ with $D \in \mathcal{D}_C$.*

This characterizes the elements of $\mathcal{C}(\theta)$ as the recurrent sets corresponding to different levels of exit.

LEMMA 8: *Let $C \in \mathcal{C}(\theta)$.*

1. *Let $D \subset C$, and $g \in G_D^{\min}$. All paths of g end up in C.*
2. *Let $g \in G_C^{\min}$. There is a unique $s \in C$, such that $g(s) \notin C$.*

Proof: Since $\mathbf{Q}_\theta(C|D) = 1$, one has $G_D^{\min}(s \to t) = \emptyset$, for each $s \in D$, $t \notin C$. This proves the first part of the lemma. Let $g \in G_C^{\min}$, and let $s_0 \in C$ be a state such that $g(s_0) \notin C$. Given any $C\backslash\{s_0\}$-graph $\widetilde{g}_{C\backslash\{s_0\}}$, the union $\widetilde{g}_{C\backslash\{s_0\}} \cup \{(s_0, g(s_0))\}$

is a C-graph. Since $g \in G_C^{\min}$, the restriction $\bar{g}$ of g to $C\backslash\{s_0\}$ thus belongs to $G^{\min}_{C\backslash\{s_0\}}$. Since $\mathbf{Q}_\theta(s_0|C\backslash\{s_0\}) = 1$, all paths of $\bar{g}$ lead to s_0. In particular, $g(s) \in C$, for each $s \in C\backslash\{s_0\}$. ∎

We conclude this section with few useful properties. The next result is a consequence of the fact that, for $C \in \mathcal{C}(\theta)$, no strict subset of $\mathcal{D}_C$ is closed under exit distributions

LEMMA 9: *Let $C \in \mathcal{C}(\theta)$, and let $\mathcal{D}_C$ be the collection of its sons. Let $\mathcal{D}'$ be a strict subset of $\mathcal{D}_C$, and $g \in G^{\min}_{\cup_{D\in\mathcal{D}'}D}$. For each $D \in \mathcal{D}'$, the restriction g_D of g to D belongs to $G_D^{\min}$.*

Proof: It is enough to construct, for each $D \in \mathcal{D}'$, a D-graph $\bar{g}_D \in G_D^{\min}$ such that $\cup_{D\in\mathcal{D}'}\bar{g}_D$ is a $\cup_{D\in\mathcal{D}'}D$-graph. Let $\widehat{p}$ be the transition function defined on $\mathcal{D}_C$ by the exit distributions: $\widehat{p}(D'|D) = \mathbf{Q}_\theta(D'|D)$, for $D, D' \in \mathcal{D}_C$. By Lemma 7, $\widehat{p}$ is irreducible. Therefore, there exists a $\mathcal{D}'$-graph γ such that $\widehat{p}(\gamma(D)|D) > 0$ for each edge $(D, \gamma(D))$ of γ. For $D \in \mathcal{D}'$, choose $t \in \gamma(D)$ such that $\mathbf{Q}_\theta(t|D) > 0$, and $\bar{g}_D \in G_D^{\min}(s \to t)$ (where $s \in D$ is arbitrary). By construction, $\cup_{D\in\mathcal{D}'}\bar{g}_D$ is a $\cup_{D\in\mathcal{D}'}D$-graph. ∎

LEMMA 10: *Let $C, D, E \in \bar{\mathcal{C}}(\theta)$ be given, with $C \subsetneq D \subset E$. Let $g_{E\backslash D} \in G^{\min}_{E\backslash D}$, $g_{D\backslash C} \in G^{\min}_{D\backslash C}$. The union $g_{E\backslash D} \cup g_{D\backslash C}$ belongs to $G_{E\backslash C}$.*

Proof: All paths of $g_{D\backslash C}$ end up in C, hence $g_{E\backslash D} \cup g_{D\backslash C}$ has no cycle. ∎

1.3 MARKOV CHAINS WITH PUISEUX TRANSITION FUNCTIONS. In the literature on large deviations and simulated annealing algorithms, a one-parameter family (p_ε) of transition probabilities is given, and one studies its asymptotic behavior, as ε goes to zero. It is assumed that, for each $s, s' \in S$, $p_\varepsilon(s'|s) \sim \pi(s,s')\varepsilon^{d(s,s')}$ for some $\pi(s,s'), d(s,s') \geq 0$.

We relate the previous section to this assumption, and derive a few auxiliary properties.

Let $(p_\varepsilon)_{\varepsilon>0}$ be a family of elements of $\mathcal{O}$, indexed by $\varepsilon > 0$. Assume that, for each edge (s, s') of Γ,

$$\lim_{\varepsilon\to 0} \frac{p_\varepsilon(s'|s)}{\pi(s,s')\varepsilon^{d(s,s')}} = 1, \quad \text{for some } \pi(s,s') > 0,\ d(s,s') \geq 0.$$

Clearly,

$$\theta(g_1, g_2) = \lim_{n\to+\infty} \frac{w_{p_\varepsilon}(g_1)}{w_{p_\varepsilon}(g_2)} = \begin{cases} 0 & \text{if } d(g_1) > d(g_2) \\ \dfrac{\prod_{(s,s')\in g_1}\pi(s,s')}{\prod_{(s,s')\in g_2}\pi(s,s')} & \text{if } d(g_1) = d(g_2) \\ +\infty & \text{if } d(g_1) < d(g_2) \end{cases}$$

where $d(g) = \sum_{(s,s')\in g} d(s,s')$ for $g \in \mathcal{G}$.

The next result shows that there is essentially no loss of generality in considering only this type of transition.

LEMMA 11: *Let (p_n) be a sequence in $\mathcal{O}$ such that*

$$\theta(g_1, g_2) = \lim_{n\to\infty} \frac{w_{p_n}(g_1)}{w_{p_n}(g_2)}$$

exists for each $g_1, g_2 \in \mathcal{G}$. There exist $\varepsilon_0 > 0$ and a family $(p_\varepsilon)_{\varepsilon<\varepsilon_0}$ such that the following two properties hold:

1. $\lim_{\varepsilon\to 0} \frac{w_{p_\varepsilon}(g_1)}{w_{p_\varepsilon}(g_2)} = \theta(g_1, g_2)$ *for each g_1, g_2;*
2. *for each edge (s, s') of Γ, the function $\varepsilon \mapsto p_\varepsilon(s'|s)$ has an expansion in Puiseux series on $(0, \varepsilon_0)$:*

$$p_\varepsilon(s'|s) = \sum_{k=0}^{+\infty} a_k \varepsilon^{k/M}, \quad \text{for some real numbers } (a_k)_{k\in\mathbf{N}} \text{ and } M \in \mathbf{N}.$$

Proof: For $\varepsilon > 0$, define K_ε as the set of vectors $p \in \mathbf{R}^{S\times S}$ such that:

1. $p \in \mathcal{O}$;
2. $\begin{cases} \prod_{(s,s')\in g_1} p(s'|s) \le \varepsilon \prod_{(s,s')\in g_2} p(s'|s) & \text{if } \theta(g_1,g_2) = 0 \\ \theta(g_1,g_2) - \varepsilon \le \frac{\prod_{(s,s')\in g_1} p(s'|s)}{\prod_{(s,s')\in g_2} p(s'|s)} \le \theta(g_1,g_2) + \varepsilon & \text{if } 0 < \theta(g_1,g_2) < +\infty \end{cases}$

for every $g_1, g_2 \in \mathcal{G}$.

The set K_ε is non-empty for $\varepsilon > 0$ small enough. The set $\{(\varepsilon, p), \varepsilon > 0, p \in K_\varepsilon\}$ is defined by a finite number of polynomial equalities and inequalities, i.e. is a real semialgebraic set. The result follows by repeating the proof of Lemma VII 2.2, p. 394 in [5]. ∎

We define the **valuation** of $C \subseteq S\backslash S^*$ as

$$d_C = \min_{g\in G_C} d(g)$$

(and $d_\emptyset = 0$). We interpret some of the previous results in terms of valuation, and give additional results. First, observe that $G_C^{\min} = \{g \in G_C, d(g) = d_C\}$.

LEMMA 12: *Let A and B be disjoint subsets of $S\backslash S^*$. One has $d_{A\cup B} \ge d_A + d_B$.*

Proof: For each $g \in G_{A\cup B}$, the restrictions g_A and g_B to A and B belong respectively to G_A and G_B, and

$$d(g) = d(g_A) + d(g_B).$$

The result follows by taking the infimum over g. ∎

The next result, a consequence of Lemma 9, gives a condition under which equality holds.

LEMMA 13: *Let $C \in \mathcal{C}(\theta)$, $\mathcal{D}_C$ be the collection of its sons, and $\mathcal{D}' \subset \mathcal{D}_C$ be given. One has*

$$d_{\cup_{D\in\mathcal{D}'}D} = \sum_{D\in\mathcal{D}'} d_D.$$

The next result is a corollary of Lemma 10.

LEMMA 14: *Let $C, D, E \in \bar{\mathcal{C}}(\theta)$ be given, with $C \subseteq D \subset E$. One has*

$$d_{E\setminus C} = d_{E\setminus D} + d_{D\setminus C}.$$

In the sequel, we let $s_0 \in S\setminus S^*$, and $\delta > 0$ be given. We define D as the minimal element of $\mathcal{C}(\theta)$ such that

$$s_0 \in D \quad \text{and} \quad d_D > d_{D\setminus\{s_0\}} + \delta|D|. \tag{4}$$

We assume that such a set exists.

LEMMA 15: *Let $C \in \bar{\mathcal{C}}(\theta)$, with $C \subset D$. Assume $s_0 \in C$. Then $d_C \leq d_{C\setminus\{s_0\}} + \delta|C|$.*

Proof: The claim is clear if $C \in \mathcal{C}(\theta)$, by the minimality of D. Otherwise, C is a transient state, hence $d_C = 0$. ∎

LEMMA 16: *For every $s_1 \in D$, one has*

$$d_{D\setminus\{s_1\}} \leq d_{D\setminus\{s_0\}} + \delta(|D| - 1).$$

Proof: Let $s_1 \in D$. If $s_0 = s_1$, there is nothing to prove. We thus assume $s_1 \neq s_0$. Let $C \subseteq D$ be the least element of $\mathcal{C}(\theta)$ that contains both s_0 and s_1. Let $\mathcal{D}_C$ be the collection of the sons of C, and $C_0 \subset C$ be the son that contains s_0.

It cannot be the case that $s_1 \in C_0$. Otherwise indeed, C_0 would contain at least two states, s_0 and s_1, hence would belong to $\mathcal{C}(\theta)$—a contradiction to the definition of C. By Lemma 14 (twice), $d_{D\setminus\{s_1\}} = d_{D\setminus C} + d_{C\setminus\{s_1\}}$ and $d_{C\setminus\{s_1\}} = d_{C\setminus(C_0\cup\{s_1\})} + d_{C_0}$. By Lemma 12, $d_{C_0} \leq d_{C_0\setminus\{s_0\}} + \delta|C_0|$, therefore

$$\begin{aligned} d_{D\setminus\{s_1\}} &\leq d_{D\setminus C} + d_{C\setminus(C_0\cup\{s_1\})} + d_{C_0\setminus\{s_0\}} + \delta|C_0| \\ &\leq d_{D\setminus\{s_0,s_1\}} + \delta|C_0|, \end{aligned}$$

where the last inequality follows from Lemma 12, applied twice. The result follows since C_0 is a strict subset of D. ∎

COROLLARY 17: *Let (s, s') be an edge of Γ with $s \in D$, $s' \notin D$. Then $d(s, s') > \delta$.*

Proof: The union of $\{(s, s')\}$ and of a graph in $G^{\min}_{D\setminus\{s\}}$ is a D-graph g such that

$$\begin{aligned} d(g) &= d_{D\setminus\{s\}} + d(s, s') \\ &\leq d_{D\setminus\{s_0\}} + \delta(|D| - 1) + d(s, s'), \end{aligned}$$

using Lemma 16. The result follows by (4). ∎

2. Stochastic games

2.1 BASIC RESULTS. We apply here the results of Section 1 to stochastic games. For convenience, we restrict ourselves to two-player games, though the generalization is straightforward.

A two-player stochastic game is described by: (i) a finite set S of states, (ii) finite sets A and B of actions available to players 1 and 2 respectively, (iii) a transition function q that specifies, for each $(s, a, b) \in S \times A \times B$, a probability distribution $q(\cdot|s, a, b)$ over S, and (iv) a payoff function which here is irrelevant.

The game is played in stages. At stage $k \in \mathbf{N}$, the two players know both the past play and the current state s_k; they independently choose actions $a_k \in A$, $b_k \in A$, possibly at random. The next state is drawn according to $q(\cdot|s_k, a_k, b_k)$, and the game proceeds to stage $k + 1$.

Given a finite set M, we denote by $\Delta(M)$ the set of probability distributions over M.

The behavior of a player is characterized by a **strategy**, which prescribes, for each finite history, a distribution over the set of actions. A strategy that depends only on the current state is called **stationary**. A stationary strategy of player 1 can be identified with a vector $x = (x_s)_{s\in S}$, where $x_s \in \Delta(A)$ is the distribution according to which player 1 chooses an action, whenever the current state is s. Similarly, a stationary strategy of player 2 will be identified with a vector $y = (y_s)_{s\in S}$, where $y_s \in \Delta(B)$. We denote by $\mathcal{O}_1$ the set of stationary strategies $x \in \Delta(A)^S$ with full support: $x_s(a) > 0$ for every $(s, a) \in S \times A$. The set $\mathcal{O}_2$ of stationary strategies of player 2 with full support is defined similarly.

Given an initial state $s \in S$, any pair (x, y) of stationary strategies induces a probability distribution $\mathbf{P}_{s,x,y}$ over the space $(S \times A \times B)^{\mathbf{N}}$ of plays. The coordinate process $(s_k, a_k, b_k)_{k\geq 1}$ follows a Markov chain under $\mathbf{P}_{s,x,y}$.

A state $s \in S$ is **absorbing** if $q(s|s,a,b) = 1$ for every $(a,b) \in A \times B$. The subset of absorbing states is denoted by S^*. We make the following assumptions:

A1 $S^* \neq \emptyset$;

A2 for every initial state $s \in S$, and $(x,y) \in \mathcal{O}_1 \times \mathcal{O}_2$, the game reaches S^* in finite time, $\mathbf{P}_{s,x,y}$-a.s.

In [9], it is proven that, in order to prove the existence of equilibrium payoffs in two-player games, it suffices to consider games that satisfy **A1** and **A2**.

In order to apply the results of the previous section, we introduce the graph Γ over S, defined by the condition

$$(s,s') \text{ is an edge of } \Gamma \Leftrightarrow q(s'|s,a,b) > 0 \text{ for some } (a,b) \in A \times B.$$

By **A1** and **A2**, the set of sinks for Γ coincides with S^* and, for each $s \in S$, there exists a path from s to S^*. We therefore may apply the results of section 1.

Given $(x,y) \in \mathcal{O}_1 \times \mathcal{O}_2$, the transition function $p_{x,y}$ on S defined by

$$p_{xy}(s'|s) = \sum_{(a,b)\in A\times B} x_s(a) y_s(b) q(s'|s,a,b)$$

belongs to the set $\mathcal{O}$ associated with the graph Γ. In that sense, $\mathcal{O}_1 \times \mathcal{O}_2$ is embedded in the set Θ defined in Remark 2. For convenience, we write $\phi(x,y)$ instead of $\phi(p_{xy})$. The proofs can be extended to prove the next result.

LEMMA 18: *Let $(x_n,y_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{O}_1 \times \mathcal{O}_2$ such that $\theta = \lim_{n\to+\infty} \phi(x_n,y_n)$ exists. Then:*

- *$(x^\theta, y^\theta) = \lim_{n\to+\infty}(x_n,y_n)$ exists.*
- *There exists $\varepsilon_0 > 0$ and a family $(x^\varepsilon, y^\varepsilon)_{\varepsilon<\varepsilon_0}$ of elements of $\mathcal{O}_1 \times \mathcal{O}_2$ such that the following two properties hold:*
  - *$\lim_{\varepsilon\to 0} \phi(x^\varepsilon, y^\varepsilon) = \theta$;*
  - *for each $(s,a,b) \in S\backslash S^* \times A \times B$, the functions $\varepsilon \mapsto x_s^\varepsilon(a)$ and $\varepsilon \mapsto y_s^\varepsilon(b)$ have an expansion in Puiseux series on $(0,\varepsilon_0)$.*

Hence, for each $(s,a,b) \in S\backslash S^* \times A \times B$, there exist positive numbers $\pi_s(a)$, $\pi_s(b), d_s(a), d_s(b)$ such that $x_s(a) \sim \pi_s(a)\varepsilon^{d_s(a)}$ and $y_s(b) \sim \pi_s(b)\varepsilon^{d_s(b)}$ at zero.

In the sequel, we let $D \in \mathcal{C}(\theta)$, $s_0 \in D$, and $\delta > 0$ be given. We assume that, among the communicating sets that contain s_0, D is the smallest one such that the inequality below is satisfied:

$$d_D > d_{D\backslash\{s_0\}} + \delta|D| \tag{5}$$

(we assume that such a set exists). We conclude with a result which is motivated by the analysis of positive recursive games (see Vieille [10]).

LEMMA 19: *Let F be an element of $\mathcal{C}(\theta)$ that contains D. Let $g \in G_F^{\min}$ and $\bar{g} \in G_F$ be given. Let $(s_1, g(s_1))$ be the unique edge of g whose endpoint is not in F, and $s_2 \in D$ be a state such that $\bar{g}(s_2) \notin D$. If $s_1 \in D$ and $s_1 \neq s_2$,*

$$d(s_2, \bar{g}(s_2)) > d(s_2, g(s_2)) + \delta.$$

Proof: Let C be the smallest element of $\mathcal{C}(\theta)$ that contains both s_1 and s_2, $\mathcal{D}_C$ be the collection of its sons, and $C_2 \in \mathcal{D}_C$ the son that contains s_2. We choose $\widetilde{g}_{C_2\setminus\{s_2\}} \in G^{\min}_{C_2\setminus\{s_2\}}$ and let $\widetilde{g}_{C_2}$ denote the union of $\widetilde{g}_{C_2\setminus\{s_2\}}$ and $\{(s_2, \bar{g}(s_2))\}$. Observe that $\widetilde{g}_{C_2}$ is a C_2-graph. Since $C_2 \in \bar{\mathcal{C}}(\theta)$, all paths of $\widetilde{g}_{C_2\setminus\{s_2\}}$ lead to s_2, hence all paths of $\widetilde{g}_{C_2}$ lead to $\bar{g}(s_2) \notin D$.

Since $s_1 \neq s_2$, one has $s_1 \notin C_2$, by definition of C. Denote by C_1 the son of C that contains s_1. Since $g \in G_F^{\min}$, the restriction of g to $F\setminus\{s_1\}$ belongs to $G^{\min}_{F\setminus\{s_1\}}$ (otherwise, substituting in g a graph in $G^{\min}_{F\setminus\{s_1\}}$ to the restriction of g would yield a graph $\widehat{g}$ in G_F such that $d(\widehat{g}) < d(g)$). By Lemma 14 (twice), the restriction $g_{C\setminus C_1}$ of g to $C\setminus C_1$ belongs to $G^{\min}_{C\setminus C_1}$. By Lemma 13, the restriction g_{C_2} of $g_{C\setminus C_1}$ to C_2 belongs to $G^{\min}_{C_2}$. Therefore,

$$d_{C_2} = d(g_{C_2}) \geq d_{C_2\setminus\{s_2\}} + d(s_2, g(s_2)),$$

hence

$$d(\widetilde{g}_{C_2}) = d_{C_2\setminus\{s_2\}} + d(s_2, \bar{g}(s_2)) \leq d_{C_2} + d(s_2, \bar{g}(s_2)) - d(s_2, g(s_2)). \tag{6}$$

We now discuss according to the location of s_0 in D.

CASE 1: $s_0 \notin C_2$. Let E denote the smallest element of $\mathcal{C}(\theta)$ that contains both s_0 and s_2, $\mathcal{E}$ be the collection of its sons, and $E_2 \in \mathcal{E}$ the son that contains s_2. Hence $E_2 = C_2$ if $s_0 \in C$, and $E_2 \supset C_2$ if $s_0 \notin C$. By definition of E, $s_0 \notin E_2$. Denote by E_0 the son of E that contains s_0. Choose $\widetilde{g}_{D\setminus E}$, $\widetilde{g}_{E\setminus E_2}$ and $\widetilde{g}_{E_2\setminus C_2}$ in $G^{\min}_{D\setminus E}$, $G^{\min}_{E\setminus E_2}$ and $G^{\min}_{E_2\setminus C_2}$ respectively: the paths of $\widetilde{g}_{D\setminus E}$ lead to E, the paths of $\widetilde{g}_{E\setminus E_2}$ lead to E_2 and the paths of $\widetilde{g}_{E_2\setminus C_2}$ lead to C_2. Since the paths of $\widetilde{g}_{C_2}$ lead to $\bar{g}(s_2) \notin D$, the union $\widetilde{g} = \widetilde{g}_{D\setminus E} \cup \widetilde{g}_{E\setminus E_2} \cup \widetilde{g}_{E_2\setminus C_2} \cup \widetilde{g}_{C_2}$ belong to G_D, hence

$$\begin{aligned} d_D &\leq d(\widetilde{g}_{D\setminus E}) + d(\widetilde{g}_{E\setminus E_2}) + d(\widetilde{g}_{E_2\setminus C_2}) + d(\widetilde{g}_{C_2}) \\ &\leq d_{D\setminus E} + d_{E\setminus E_2} + d_{E_2\setminus C_2} + d_{C_2} + d(s_2, \bar{g}(s_2)) - d(s_2, g(s_2)). \end{aligned} \tag{7}$$

By Lemma 13, $d_{E\setminus E_2} = \sum_{E' \in \mathcal{E},\ E' \neq E_2} d_{E'}$. By Lemma 15, $d_{E_0} \leq d_{E_0\setminus\{s_0\}} + \delta|E_0|$.

By substitution into (7), using Lemma 12, one gets

$$\begin{aligned} d_D \leq & d_{D\setminus E} + d_{E_0\setminus\{s_0\}} + \sum_{E'\in\mathcal{E}\setminus\{E_0,E_2\}} d_{E'} + d_{E_2\setminus C_2} + d_{C_2} + \delta|E_0| \\ & + d(s_2,\bar{g}(s_2)) - d(s_2,g(s_2)) \\ \leq & d_{D\setminus\{s_0\}} + \delta|E_0| + d(s_2,\bar{g}(s_2)) - d(s_2,g(s_2)). \end{aligned}$$

Since $E_0 \subset D$, the result follows from (5).

CASE 2: $s_0 \in C_2$. It is enough to prove that

$$d_D \leq d_{D\setminus\{s_0\}} + \delta|C_2| + d(s_2,\bar{g}(s_2)) - d(s_2,g(s_2)), \tag{8}$$

by (5) and since C_2 is a strict subset of D.

If $C_2 = \{s_0\} = \{s_2\}$ is a transient state, $d_{C_2} = d(s_2,g(s_2)) = 0$, hence $d(\widetilde{g}_{C_2}) = d(s_2,\bar{g}(s_2)) - d(s_2,g(s_2))$. The union $\widetilde{g}_D$ of $\widetilde{g}_{C_2}$ and of any graph in $G^{\min}_{D\setminus\{s_0\}}$ is a D-graph, with degree

$$d(\widetilde{g}_D) = d_{D\setminus\{s_0\}} + d(s_2,\bar{g}(s_2)) - d(s_2,g(s_2)). \tag{9}$$

Since $d(\widetilde{g}_D) \geq d_D$, the inequality (8) follows from (9).

Otherwise, $C_2 \in \mathcal{C}(\theta)$ and C_2 is a strict subset of D. By Lemma 15, $d_{C_2} \leq d_{C_2\setminus\{s_0\}} + \delta|C_2|$. Since all paths of $\widetilde{g}_{C_2}$ lead to $\bar{g}(s_2) \notin F$, the union $\widetilde{g}_D$ of $\widetilde{g}_{C_2}$ and of any graph in $G^{\min}_{D\setminus C_2}$ is a D-graph that satisfies

$$\begin{aligned} d(\widetilde{g}_D) &= d_{D\setminus C_2} + d(\widetilde{g}_{C_2}) \\ &\leq d_{D\setminus C_2} + d_{C_2} + d(s_2,\bar{g}(s_2)) - d(s_2,g(s_2)) \\ &\leq d_{D\setminus C_2} + d_{C_2\setminus\{s_0\}} + \delta|C_2| + d(s_2,\bar{g}(s_2)) - d(s_2,g(s_2)) \\ &\leq d_{D\setminus\{s_0\}} + \delta|C_2| + d(s_2,\bar{g}(s_2)) - d(s_2,g(s_2)), \end{aligned}$$

where the first inequality uses (6), the second follows from Lemma 15, and the last one from Lemma 12. Hence (8) holds. ∎

The next result complements the previous one, by giving a result in the case $s_1 = s_2$.

LEMMA 20: *Let F be an element of $\mathcal{C}(\theta)$ that contains D. Let $g \in G^{\min}_F$ be given, $(s_1, g(s_1))$ be the unique edge of g whose endpoint is not in F, and $(a_1,b_1) \in A\times B$ the unique pair such that $q(g(s_1)|s_1,a_1,b_1) > 0$. Assume that $d_{s_1}(a_1) = 0$. Let $\bar{g}$ be an F-graph, such that $\bar{g}(s_1) \notin D$, and $(\bar{a}_1,\bar{b}_1) \in A\times B$ the unique pair such that $q(\bar{g}(s_1)|s_1,\bar{a}_1,\bar{b}_1) > 0$. One has*

$$d_{s_1}(\bar{b}_1) \leq \delta \Rightarrow d_{s_1}(\bar{a}_1) > 0.$$

Proof: The union of any graph in $G^{\min}_{D\setminus\{s_1\}}$ and of $\{(s_1,\bar{g}(s_1))\}$ is a D-graph g_D with valuation

$$d(g_D)\leq d_{D\setminus\{s_1\}}+d_{s_1}(\bar{a}_1)+\delta\leq d_{D\setminus\{s_0\}}+\delta(|D|-1)+d_{s_1}(\bar{a}_1)+\delta,$$

where the second inequality uses Lemma 16. The result follows, using (5). ∎

COROLLARY 21: *Let $F\in\mathcal{C}(\theta)$ such that $D\subseteq F$. Let $g,\bar{g}\in G^{\min}_F$. Let (s_1,s_1') be the unique edge of g such that $s_1'\notin F$. Let (s_2,s_2') be an edge of $\bar{g}$ such that $s_2\in D$ and $s_2'\notin D$. Assume that $s_1\in D$, and that $d(s_1,a_{s_1}(g))=0$. Then*

$$d(s_2,b_{s_2}(\bar{g}))\leq\delta\Rightarrow d(s_2,a_{s_2}(\bar{g}))>d(s_2,a_{s_2}(g)).$$

Proof: If $s_1\neq s_2$, $d(s_2,b_{s_2}(\bar{g}))+d(s_2,a_{s_2}(\bar{g}))>d(s_2,b_{s_2}(g))+d(s_2,a_{s_2}(g))+\delta$ by Lemma 19.

If $s_1=s_2$, one has $d(s_2,a_{s_2}(g))=d(s_1,a_{s_1}(g))=0$ and $d(s_1,b_{s_1}(\bar{g}))\leq\delta\Rightarrow d(s_1,a_{s_1}(\bar{g}))>0$ by Lemma 20. ∎

2.2 CONCLUDING RESULTS. Let $C\in\mathcal{C}(\theta)$. We give a link between $\mathbf{Q}_\theta(\cdot|C)$ and the exit distributions induced by perturbations of (x^θ,y^θ). Denote by

$$\widetilde{e}_C=1+\{n\geq 1,q(C|s_n,a_n,y_n)<1\}$$

the stage following the first one in which an action combination is played that might induce exit from C. Denote by $\widetilde{\mathbf{Q}}_{s,x_n,y_n}(\cdot|C)$ the law of $\widetilde{e}_C$ under (x_n,y_n) starting from s, and set

$$\widetilde{\mathbf{Q}}_\theta(\cdot|C)=\lim_{n\to+\infty}\widetilde{\mathbf{Q}}_{s,x_n,y_n}(\cdot|C).$$

The support of a probability distribution μ is denoted by $\operatorname{Supp}\mu$. We define

$$\mathcal{Q}^1_C(x^\theta,y^\theta)=\{q(\cdot|s,a,y^\theta_s),\text{ where }(s,a)\in C\times A\text{ and }q(C|s,a,y^\theta_s)<1\},$$
$$\mathcal{Q}^2_C(x^\theta,y^\theta)=\{q(\cdot|s,x^\theta_s,b),\text{ where }(s,b)\in C\times B\text{ and }q(C|s,x^\theta_s,b)<1\},$$
$$\mathcal{Q}^j_C(x^\theta,y^\theta)=$$
$$\{q(\cdot|s,a,b),\text{ where }q(C|s,a,b)=0,\ q(C|s,a,y^\theta_s)=q(C|s,x^\theta_s,b)<1\}.$$

LEMMA 22: *The distribution $\mathbf{Q}_\theta(\cdot|C)$ belongs to the convex hull of $\mathcal{Q}^1_C(x^\theta,y^\theta)\cup\mathcal{Q}^2_C(x^\theta,y^\theta)\cup\mathcal{Q}^j_C(x^\theta,y^\theta)$.*

Proof: For $s\in C$, define $(A\times B)(s)=\{(a,b)\in A\times B,\ q(C|s,a,b)<1\}$. For $t\in S$, one has

$$\widetilde{\mathbf{Q}}_\theta(t|C)=\lim_{\varepsilon\to 0}\frac{\sum_{s\in C}\{\sum_{A\times B(s)}x^\varepsilon_s(a)y^\varepsilon_s(b)q(t|s,a,b)\}\times\{\sum_{g\in G_{C\setminus\{s\}}}w_{p_\varepsilon}(g)\}}{\sum_{g\in G_C}w_{p_\varepsilon}(g)}.$$

Thus

$$\widetilde{\mathbf{Q}}_\theta(\cdot|C) = \sum_{s\in C}\sum_{A\times B(s)} \alpha_{sab} q(\cdot|s,a,b),$$

where

$$\alpha_{sab} = \lim_{\varepsilon\to 0} \frac{x_s^\varepsilon(a) y_s^\varepsilon(b)\{\sum_{g\in G_{C\setminus\{s\}}} w_{p_\varepsilon}(g)\}}{\sum_{g\in G_C} w_{p_\varepsilon}(g)}.$$

Observe now that, given (s,a) such that $q(C|s,a,y_s^\theta) < 1$, and $b_1, b_2 \in \operatorname{Supp} y_s^\theta$, one has

$$\frac{\alpha_{sab_1}}{\alpha_{sab_2}} = \frac{y_s^\theta(b_1)}{y_s^\theta(b_2)}.$$

The same property holds if one exchanges the two players. Observe also that, given $a \notin \operatorname{Supp} x_s^\theta$ and $b \notin \operatorname{Supp} y_s^\theta$, such that $q(C|s,a,y_s^\theta) < 1$ or $q(C|s,x_s^\theta,b) < 1$, one has $\alpha_{sab} = 0$. This suffices to yield the result. ∎

We now define an alternative notion of communication.

A perturbation of μ is a distribution $\widetilde{\mu}$ such that $\operatorname{Supp}\mu \subseteq \operatorname{Supp}\mu'$.

Given any pair (x,y) of stationary strategies and $C \subseteq S\backslash S^*$, we define a directed graph $G_C(x,y)$ as follows:

- the set of vertices is C;
- for any two states $s, s' \in C$, there is an edge from s to s' if and only if there exist perturbations $\widetilde{x}_s$ and $\widetilde{y}_s$ of x_s and y_s respectively such that $q(s'|s,\widetilde{x}_s,\widetilde{y}_s) > 0$ while $q(C|s,\widetilde{x}_s,\widetilde{y}_s) = 1$.

Definition 23: Let (x,y) be a pair of stationary strategies. A set $C \subseteq S\backslash S^*$ **communicates under** (x,y) if the graph $G_C(x,y)$ is strongly connected.

LEMMA 24: *If $C \in \mathcal{C}(\theta)$, then C communicates under (x^θ, y^θ).*

Proof: We proceed by induction on the size of C. If $|C| = 1$, C is a closed singleton set under (x^θ, y^θ), hence the result holds. Let $C \in \mathcal{C}(\theta)$ and $\mathcal{D}_C$ be the collection of its sons. Let $s_1, s_2 \in C$ be given. We need to prove that $G_C(x^\theta, y^\theta)$ contains a path from s_1 to s_2. By the induction hypothesis, this is true if s_1 and s_2 belong to the same son of C.

Assume now that $s_1 \in D_1$ and $s_2 \in D_2$, where D_1 and D_2 are distinct sons of C. We prove that $G_C(x^\theta, y^\theta)$ contains a path from s_1 to s_2. W.l.o.g., we assume that $\mathbf{Q}_\theta(D_2|D_1) > 0$. By Lemma 22, there exists $q \in \mathcal{Q}^1_{D_1}(x^\theta,y^\theta) \cup \mathcal{Q}^2_{D_1}(x^\theta,y^\theta) \cup \mathcal{Q}^j_{D_1}(x^\theta,y^\theta)$ such that $q(D_2) > 0$ and $q(C) = 1$. To fix the ideas, assume $q \in \mathcal{Q}^1_{D_1}(x^\theta,y^\theta)$, and choose $(\bar{s}_1, a_1, \bar{s}_2) \in D \times A \times \bar{D}$ such that $p(\bar{s}_2|\bar{s}_1, a_1, y^\theta_{\bar{s}_1}) > 0$ and $p(C|\bar{s}_1, a_1, y^\theta_{\bar{s}_1}) = 1$. In particular, the edge $(\bar{s}_1, \bar{s}_2)$ belongs to $G_C(x^\theta, y^\theta)$.

By the first observation, $G_C(x^\theta, y^\theta)$ contains a path from s_1 to $\bar{s}_1$ and from $\bar{s}_2$ to s_2.

By Lemma 7, $\mathcal{D}_C$ is a recurrent set for $\widetilde{p}$ defined by $\widetilde{p}(D'|D) = \mathbf{Q}_\theta(D'|D)$. The result follows. ∎

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